

Profinite completion of Grigorchuk's group is not finitely presented

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Abstract

In this paper we prove that the profinite completion $\hat{\mathcal{G}}$ of the Grigorchuk group \mathcal{G} is not finitely presented as a profinite group. We obtain this result by showing that $H^2(\hat{\mathcal{G}}, \mathbb{F}_2)$ is infinite dimensional. Also several results are proven about the finite quotients $\mathcal{G}/St_{\mathcal{G}}(n)$ including minimal presentations and Schur Multipliers.

In 1980's R. Grigorchuk constructed groups of automorphisms of rooted trees having extraordinary properties. One prototypical example (usually referred as the *first Grigorchuk Group* and will be denoted by \mathcal{G} throughout the paper) was the object of study of several researchers in the last 30 years. It has many interesting properties some of which can be summarized as follows: It is an example of a Burnside group i.e. a finitely generated infinite periodic group. It's growth function has intermediate growth rate and it is the first example of such a group. It is an amenable group which is not elementary amenable. It is not finitely presented but has a special kind of recursive presentation. It is a just infinite branch group i.e. an infinite group whose nontrivial quotients are all finite and its lattice of normal subgroups resembles the binary rooted tree. We refer to [8], [7], [13], [4] for generalities about the Grigorchuk group and related topics.

As \mathcal{G} is an interesting group from the point view of discrete groups, its profinite completion $\hat{\mathcal{G}}$ has several interesting properties in the class of profinite groups. First of all it coincides with the closure of \mathcal{G} in the full automorphism group of the binary rooted tree. It is a just-infinite profinite branch group. It has finite width (i.e. the lower central factors have bounded rank). Also it has an interesting universal property that it contains a copy of every

countably based pro-2 group. In [2] it was shown that $\hat{\mathcal{G}}$ is a counterexample to a conjecture about just-infinite pro-p groups of finite width.

Our main result is about finite presentability of $\hat{\mathcal{G}}$ as a profinite group:

Theorem. *The profinite completion $\hat{\mathcal{G}}$ of \mathcal{G} is a 3 generated pro-2 group and is not finitely presented as a profinite group.*

The main theorem is proved by showing that the cohomology group $H^2(\hat{\mathcal{G}}, \mathbb{F}_2)$ is infinite dimensional. Naturally, to achieve this goal we prove various intermediate results related to finite quotients of \mathcal{G} . A step-by-step scheme can be summarized as follows:

1. Finding presentations for the finite quotients $\mathcal{G}_n = \mathcal{G}/St_{\mathcal{G}}(n)$ (Theorem 1).
2. Using these presentations to compute Schur Multiplier $H_2(\mathcal{G}_n, \mathbb{Z}) = (C_2)^{2n-2}$ and $H^2(\mathcal{G}_n, \mathbb{F}_2) = (C_2)^{2n+1}$ (Theorem 2).
3. Using theorem 2 and the fact that \mathcal{G} is a regular branch group showing that $H^2(\hat{\mathcal{G}}, \mathbb{F}_2)$ is infinite dimensional (Theorem 5).

Also as byproduct, in theorems 3 and 4, we show that relators of the presentations from Theorem 1 are independent and find minimal presentations for the finite quotients \mathcal{G}_n .

The paper is organized as follows:

In section 1 we give basic definitions and properties of self-similar groups and specifically of \mathcal{G} . Section 2 is devoted to the discussion of main results. Last section contains the proofs together with intermediate lemmas.

Notation:

$[g, h]$ will always denote the element $g^{-1}h^{-1}gh$. h^g is used for the conjugate $g^{-1}hg$. If G is a group and S is a subset, $\langle S \rangle_G$ denotes the subgroup generated by S and $\langle S \rangle_G^\#$ denotes the normal subgroup generated by S . C_n denotes the cyclic group of order n and \mathbb{F}_p denotes the finite field of order p . G^n and G^∞ denote the n fold and infinite direct product of G with itself respectively. \hat{G} and \hat{G}_p denote profinite and pro-p completions respectively.

1 Preliminaries

Automorphisms of rooted trees

Let $X = \{0, \dots, d-1\}$ be an alphabet of d elements. Then the set X^* of finite sequences over X has the structure of a regular d -ary rooted tree. At the root one has the empty string denoted by \emptyset , and each word $w \in X^*$ has d children $\{wx \mid x \in X\}$. We will make no distinction between the set X^* and the tree it describes. The ambient object is the group of graph automorphisms of X^* denoted by $Aut(X^*)$. These are bijections on X^* which preserve incidence of vertices (i.e. prefixes in X^*). By the level of a vertex w we mean its length as a word (equivalently its distance to the root), we denote the level by $|w|$. It is easy to see that any such automorphism fixes the root and permutes vertices in the same level.

Definition. *Given a subgroup G of $Aut(X^*)$, for each $n \geq 1$ we have a normal subgroup*

$$St_G(n) = \{g \in G \mid g(w) = w \text{ for all } w \in X^* \text{ with } |w| = n\}$$

called the n -th level stabilizer of G .

Given a group G of tree automorphisms we will denote $G/St_G(n)$ by G_n .

Since each level of the tree has finitely many elements and automorphisms do not change levels, it follows that the index $[G : St_G(n)]$ is always finite. Also since $\cap St_G(n)$ is trivial, any subgroup of $Aut(X^*)$ is residually finite.

The d -ary rooted tree X^* is a self-similar geometric object. The subtree wX^* hanging down at a vertex $w \in X^*$ (i.e. words starting with w) is canonically isomorphic to the whole tree X^* via the isomorphism

$$\begin{aligned} \phi_w : X^* &\longrightarrow wX^* \\ v &\longmapsto wv \end{aligned}$$

This self-similarity also reflects upon the automorphism group:

Given an automorphism $f \in Aut(X^*)$ and a vertex $v \in X^*$ we have another automorphism denoted by f_v (called *the section of f at v*) which is uniquely determined by the equation

$$f(vw) = f(v)f_v(w) \quad \text{for all } v, w \in X^*$$

(Or equivalently $f_v(w) = \phi_{f(v)}^{-1}f(\phi_v(w))$).

Definition. A subgroup G of $\text{Aut}(X^*)$ is called *self-similar* if it is closed under taking sections of its elements i.e. for all $f \in G$ and for all $v \in X^*$ we have $f_v \in G$.

If G is self-similar we have an embedding of G into the semi-direct product

$$\begin{array}{ccc} G & \longrightarrow & (G \times \dots \times G) \rtimes S_d \\ f & \mapsto & (f_0, \dots, f_{d-1}) \quad \sigma_f \end{array}$$

where S_d is the symmetric group on d letters and $\sigma_f \in S_d$ is the permutation determined by f on the first level of the tree. f_0, \dots, f_{d-1} determine how f acts on the first level subtrees and σ_f determines how these subtrees are permuted. This semi-direct product is called a *permutational wreath product* and is usually denoted by $G \wr S_d$.

An easy way to create a self-similar group is to start with a set of symbols $\{f_0, \dots, f_k\}$ and look at the system

$$\begin{array}{ccc} f_0 & = (f_{00}, \dots, f_{0d-1}) & \sigma_0 \\ f_1 & = (f_{10}, \dots, f_{1d-1}) & \sigma_1 \\ & \dots & \dots \\ f_k & = (f_{k0}, \dots, f_{kd-1}) & \sigma_k \end{array}$$

where $f_{ij} \in \{f_0, \dots, f_k\}$ and $\sigma_i \in S_d$.

Such a system (usually referred to as a *wreath recursion*) defines a unique set of automorphisms of $\text{Aut}(X^*)$ denoted by f_0, \dots, f_k . Then we have the subgroup $G = \langle f_1, \dots, f_k \rangle_{\text{Aut}(X^*)}$ which is obviously a self-similar group since each section of each generator is again a generator. This construction is a source of many interesting groups and most of the well studied self-similar groups are defined by a wreath recursion.

In fact a self-similar group defined in this way belongs to a smaller class of self-similar groups called the *groups generated by finite automata*. For more on groups generated by automata see [12].

If G is a self-similar group acting on the d -ary tree, we have a monomorphism

$$\begin{aligned}\varphi : St_G(1) &\longrightarrow G \times \dots \times G \\ g &\mapsto (g_0, \dots, g_{d-1})\end{aligned}$$

Definition. Let G be a level-transitive self-similar group (i.e. it acts transitively on the levels of the tree). G is called regular branch over a finite index subgroup K if

$$K \times \dots \times K \leq \varphi(K)$$

The Grigorchuk Group

Definition. Let $X = \{0, 1\}$ so that X^* is the binary rooted tree. Consider the following automorphisms given by the wreath recursion:

$$\begin{aligned}a &= (1, 1) \quad \sigma \\ b &= (a, c) \\ c &= (a, d) \\ d &= (1, b)\end{aligned}$$

The subgroup \mathcal{G} they generate is called the Grigorchuk group. (σ denotes the nontrivial element in S_2).

We will list some well known properties of \mathcal{G} which will be used throughout the paper (For proofs see [7],[4]):

- \mathcal{G} is level transitive.
- $St_{\mathcal{G}}(1) = \langle b, c, d, b^a, c^a, d^a \rangle_{\mathcal{G}}$ and we have a monomorphism

$$\begin{aligned}\varphi : St_{\mathcal{G}}(1) &\longrightarrow \mathcal{G} \times \mathcal{G} \\ b &\mapsto (a, c) \\ c &\mapsto (a, d) \\ d &\mapsto (1, b) \\ b^a &\mapsto (c, a) \\ c^a &\mapsto (d, a) \\ d^a &\mapsto (b, 1)\end{aligned}$$

- In \mathcal{G} the relations $a^2 = b^2 = c^2 = d^2 = bcd = (ad)^4 = 1$ hold.
- \mathcal{G} is an infinite 2 group.

- \mathcal{G} is a regular branch group over the subgroup $K = \langle (ab)^2 \rangle_{\mathcal{G}}^{\#}$.
- \mathcal{G} is just-infinite.
- $St_{\mathcal{G}}(3) \leq K$.
- $\mathcal{G}_1 \cong C_2$, $\mathcal{G}_2 \cong C_2 \wr C_2$, $\mathcal{G}_3 \cong C_2 \wr C_2 \wr C_2$ and $|\mathcal{G}_n| = 2^{5 \cdot 2^{n-3} + 2}$ for $n \geq 3$.
- For $n \geq 4$ the kernels of the quotient maps $q_n : \mathcal{G}_n \longrightarrow \mathcal{G}_{n-1}$ are elementary abelian 2-groups and satisfy

$$|Ker(q_n)| = 2^{5 \cdot 2^{n-4}}$$

2 Main Theorems

Presentations, Schur Multipliers and independence of relators

Immediately after discovering his group, Grigorchuk proved that it is not finitely presented. In 1985 a recursive presentation was found by I.Lysenok. He proved that \mathcal{G} has the following special recursive presentation:

Theorem. (*Lysenok [19]*) *Grigorchuk group has the presentation*

$$\langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, \sigma^i((ad)^4), \sigma^i((adacac)^4), i \geq 0 \rangle,$$

where σ is the substitution

$$\sigma = \begin{cases} a \mapsto aca \\ b \mapsto d \\ c \mapsto b \\ d \mapsto c \end{cases}$$

In [10] Grigorchuk gave a systematic way of finding similar presentations in the general case and suggested the name *L-presentations*. This scheme was later used in [14],[15] to find similar presentations for other self similar groups including iterated monodromy groups. Also in [1] Bartholdi showed that every contracting self-similar branch group has such a presentation and used his theorem to find presentations for many other well known self-similar groups.

Roughly, a finite L -presentation is a generalization of a finite presentation in which one can obtain all relations by applying finitely many free group endomorphisms to finitely many initial relators. A precise definition is as follows:

Definition. *An L -presentation (or endomorphic presentation) is an expression*

$$\langle X \mid Q \mid R \mid \Phi \rangle$$

where X is a set, Q, R are subsets of the free group $F(X)$ on the set X and Φ is a set of free group endomorphisms.

It defines the group

$$G = F(X)/N$$

where

$$N = \left\langle Q, \bigcup_{\phi \in \Phi^*} \phi(R) \right\rangle_F^\#$$

It is called a finite L -presentation if X, Q, R, Φ are all finite.

For more on L -presentations see [1].

Lysenok's presentation was later used by Grigorchuk [9] to embed \mathcal{G} into a finitely presented amenable group $\tilde{\mathcal{G}}$ which is an HNN-extension of \mathcal{G} . Since $\tilde{\mathcal{G}}$ contains \mathcal{G} it is not elementary amenable and hence amenable and elementary amenable groups do not coincide even in the class of finitely presented groups.

Our first theorem gives similar presentations for the finite quotients \mathcal{G}_n and will be proved in section 3.1.

Theorem 1 *For $n \geq 3$ we have*

$$\mathcal{G}_n = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, u_0, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle$$

where

$$u_i = \sigma^i((ad)^4), \quad v_i = \sigma^i((adacac)^4), \quad w_n = \sigma^{n-3}((ac)^4), \quad t_n = \sigma^{n-3}((abac)^4)$$

and σ is the substitution given by

$$\sigma = \begin{cases} a \mapsto aca \\ b \mapsto d \\ c \mapsto b \\ d \mapsto c \end{cases}$$

As it can be seen readily, as n grows these presentations approach to the Lysenok's presentation.

Recall that given a group G , the *Schur Multiplier* of G (denoted by $M(G)$) is the second integral homology group $H_2(G, \mathbb{Z})$. For finite groups we have the isomorphism $H_2(G, \mathbb{Z}) \cong H^2(G, \mathbb{C}^*)$. If G is given by a presentation $F/R \cong G$ where F is a free group, the Hopf's formula (obtained first by Schur for finite groups and generalized to infinite groups by Hopf) gives

$$M(G) \cong R \cap F' / [R, F]$$

Hence the abelian group $R \cap F' / [R, F]$ is independent of the presentation of the group. If the given presentation is finite (i.e. F has finite rank and R is the normal closure of finitely many elements $\{r_1, \dots, r_m\}$ in F), then it is easy to see that the abelian group $R/[R, F]$ is generated by the images of $\{r_1, \dots, r_m\}$ and hence its subgroup $R \cap F' / [R, F]$ is a finitely generated abelian group. Therefore the Schur multiplier of a finitely presented group is necessarily finitely generated. The converse of this is not true. Baumslag in [5] gave an example of a non finitely presented group with trivial multiplier. For generalities about Schur multipliers of groups see [17].

The computation of the Schur multiplier of \mathcal{G} was done by Grigorchuk:

Theorem. (Grigorchuk, [10]) $M(\mathcal{G}) \cong (C_2)^\infty$

The proof of this theorem also shows that there are no redundant relators in Lysenok's presentation:

Theorem. (Grigorchuk, [10]) *The relators in the Lysenok presentation are independent, i.e. none of the relators is a consequence of the others.*

Recently it was shown in [6] that there are finitely generated infinitely presented solvable groups which do not have independent set of relators.

Our second theorem is the computation of Schur multipliers for the finite groups \mathcal{G}_n . It relies on theorem 1 and is similar to Grigorchuk's computations done in [10] with small modifications. The proof is presented in section 3.2. It was indicated to us by L.Bartholdi that a shorter proof for Theorem 2 could be given using ideas in [3].

Theorem 2. $M(\mathcal{G}_n) \cong C_2^{2n-2}$

Similarly we also prove that the relators of the presentations of Theorem 1 are independent, again the proof will be presented in section 3.2.

Theorem 3. *The relators in the presentations of theorem 1 are independent.*

Minimality of presentations and deficiency of groups

We begin this subsection with some definitions:

Definition. *If G is a group let $d(G)$ denote the minimal number of generators of G .*

Definition. *The minimal number m such that G has a presentation*

$$G = \langle x_1, \dots, x_t \mid r_1, \dots, r_m \rangle$$

(a presentation with m relators) will be denoted by $r(G)$.

Definition. *The deficiency of G (denoted by $\text{def}(G)$) is defined to be the minimal $m - t$ such that G has a presentation with t generators and m relators.*

Definition. *A presentation $G = \langle x_1, \dots, x_t \mid r_1, \dots, r_m \rangle$ is called minimal if $t = d(G)$ and $m = r(G)$.*

The following question is open (See [16]):

Do finite groups have minimal presentations?

A stronger question is the following:

Does a finite group have a presentation realizing its deficiency with $d(G)$ number of generators?

Clearly an affirmative answer to the second question gives an affirmative answer to the first. Lubotzky [18] gave affirmative answer to the analogous question in the category of profinite groups. It was proven by Rapaport [20] that the second question has affirmative answer for nilpotent groups. Therefore the groups \mathcal{G}_n have minimal presentations. Our next results exhibits such a minimal presentation for \mathcal{G}_n which is obtained from the presentations of Theorem 1 by a simple Tietze transformation. The proof relies on the following inequality for the deficiency:

Given a presentation $G = \langle x_1, \dots, x_t \mid r_1, \dots, r_m \rangle = F/R$ of a finite group G , we have the quotient map

$$\phi : R/[R, F] \rightarrow R/(R \cap F')$$

whose kernel is the Schur multiplier $M(G)$. But $R/(R \cap F') \cong RF'/F'$ which is free abelian of rank t because R has finite index in F . Hence

$$d(M(G)) = d(R/[F, R]) - t \leq m - t$$

and since $M(G)$ does not depend on the presentation, the following inequality holds

$$0 \leq d(M(G)) \leq \text{def}(G) \leq m - t \quad (1)$$

Theorem 4. *For $n \geq 3$ we have*

$$\mathcal{G}_n = \langle a, b, c \mid a^2, b^2, c^2, (bc)^2, u_0, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle$$

where

$$u_i = \sigma^i((abc)^4), \quad v_i = \sigma^i((abcacac)^4), \quad w_n = \sigma^{n-3}((ac)^4), \quad t_n = \sigma^{n-3}((abac)^4)$$

and σ is the substitution given by

$$\sigma = \begin{cases} a & \mapsto aca \\ b & \mapsto bc \\ c & \mapsto b \end{cases}$$

and this presentation is minimal and realizes the deficiency $\text{def}(\mathcal{G}_n) = 2n - 2$.

Proof. The presentations found in theorem 1 contain the relators $d = bc$. Hence applying Tietze transformations we get the asserted presentations. By theorem 2 we have $d(M(\mathcal{G}_n)) = 2n - 2$. Using equation (1) and counting generators and relators in the above presentation we get $2n - 2 \leq \text{def}(\mathcal{G}_n) \leq 2n - 2$.

We have $\mathcal{G}_3^{ab} \cong (C_2)^3$ and \mathcal{G}_n maps onto \mathcal{G}_3 . Also $\mathcal{G}^{ab} \cong (C_2)^3$ and \mathcal{G} maps onto \mathcal{G}_n . These show that $\mathcal{G}_n^{ab} \cong (C_2)^3$ and $d(\mathcal{G}_n) = 3$. Hence the above presentation realizes the deficiency with minimal number of generators. Therefore it is necessarily minimal. \square

Profinite completion of the Grigorchuk Group

The full automorphism group $\text{Aut}(X^*)$ is a profinite group. It is the inverse limit of the system

$$\{\text{Aut}(X_n^*) \mid n \geq 1\}$$

where $\text{Aut}(X_n^*)$ denotes the automorphism group of the finite tree X_n^* consisting of the first n levels and the maps

$$\phi_n : \text{Aut}(X_n^*) \longrightarrow \text{Aut}(X_{n-1}^*)$$

are given by restriction.

Given $G \leq \text{Aut}(X^*)$ one can talk about 3 groups \hat{G} , \hat{G}_p and \bar{G} where the last one denotes the closure of G in $\text{Aut}(X^*)$. Since the Grigorchuk group \mathcal{G} is a 2-group it follows that $\hat{\mathcal{G}} = \hat{\mathcal{G}}_2$. It is also true that $\bar{\mathcal{G}}$ coincides with these groups because of the following:

Definition. A subgroup $G \leq \text{Aut}(X^*)$ is said to have the congruence property if every finite index subgroup of G contains the subgroup $St_G(n)$ for some n .

Theorem. (See [11]) \mathcal{G} has the congruence property.

Now it follows that $\hat{\mathcal{G}} \cong \bar{\mathcal{G}}$ because the congruence property shows that

$$\{St_{\mathcal{G}}(n) \mid n \geq 1\}$$

is a neighborhood basis of the identity in \mathcal{G} .

The congruence property also shows that $\hat{\mathcal{G}}$ is the inverse limit of the inverse system $\{\mathcal{G}_n, n \geq 1\}$.

Recall that a profinite group G is finitely presented (as a profinite group) if there is an exact sequence

$$1 \longrightarrow R \longrightarrow \hat{F} \longrightarrow G \longrightarrow 1$$

where F is a free group with finite rank and R is the closed normal subgroup of \hat{F} generated by some $\{r_1, \dots, r_m\} \subset R$.

Clearly if $G \cong F/R$ is a presentation of a discrete group G , then $\hat{G} \cong \hat{F}/\bar{R}$ is a profinite presentation for \hat{G} where \bar{R} denotes the closure of R in \hat{F} . Therefore profinite completions of finitely presented groups are necessarily finitely presented. It was indicated us by Lubotzky (private communication) that the converse of this statement is not true. That is there are finitely generated residually finite groups G and H with G finitely presented and H not finitely presented and \hat{G} is isomorphic to \hat{H} . Therefore one can ask whether the profinite completion of the Grigorchuk group is finitely presented or not. Our last theorem shows that it indeed is not finitely presented. It relies on the following well known fact:

Theorem. (See [22] page 242) *A finitely generated pro- p group G is finitely presented if and only if $H^2(G, \mathbb{F}_p)$ is finite.*

Theorem 5 *We have $H^2(\hat{\mathcal{G}}, \mathbb{F}_2) \cong (C_2)^\infty$ and hence $\hat{\mathcal{G}}$ is not finitely presented as a profinite group.*

The proof of theorem 5 is presented in section 3.3.

3 Proofs of Theorems

3.1 Finding Presentations for \mathcal{G}_n

This section is devoted to the proof of theorem 1.

Let $\Gamma = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4 \rangle$. Let us denote by $\pi : \Gamma \longrightarrow \mathcal{G}$ the canonical surjection. Consider the subgroup $\Xi = \langle b, c, d, b^a, c^a, d^a \rangle_\Gamma$ which is the lift of the first level stabilizer $St_{\mathcal{G}}(1)$ to Γ .

We have a homomorphism

$$\begin{aligned}\bar{\varphi}: \Xi &\longrightarrow \Gamma \times \Gamma \\ b &\mapsto (a, c) \\ c &\mapsto (a, d) \\ d &\mapsto (1, b) \\ b^a &\mapsto (c, a) \\ c^a &\mapsto (d, a) \\ d^a &\mapsto (b, 1)\end{aligned}$$

which is analogous to $\varphi : St_{\mathcal{G}}(1) \longrightarrow \mathcal{G} \times \mathcal{G}$.

(The fact that $\bar{\varphi}$ is well defined can be checked by first finding a presentation for Ξ using Reidemeister-Schreier process and checking that it maps relators to relators.)

Given $w \in \Xi$ let us write $\bar{\varphi}(w) = (w_0, w_1)$ which is consistent with the section notation of tree automorphisms.

Recall the substitution σ from theorem 1 given by

$$\sigma = \begin{cases} a \mapsto aca \\ b \mapsto d \\ c \mapsto b \\ d \mapsto c \end{cases}$$

It is easy to check that given $w \in \Xi$ one has

$$\bar{\varphi}(\sigma(w)) = (v, w)$$

where $v \in \langle a, d \rangle_{\Gamma} \cong D_8$. Therefore since all u_i, v_i, w_i, t_i are 4-th powers and D_8 has exponent 4, we have the following equalities:

$$\begin{aligned}\bar{\varphi}(u_i) &= (1, u_{i-1}) \\ \bar{\varphi}(v_i) &= (1, v_{i-1}) \\ \bar{\varphi}(w_i) &= (1, w_{i-1}) \\ \bar{\varphi}(t_i) &= (1, t_{i-1})\end{aligned}\tag{2}$$

Let $\Omega = Ker(\pi)$ so that $\mathcal{G} = \Gamma/\Omega$. It is known (for example see [7]) that Ω is a strictly increasing union $\Omega = \bigcup_n \Omega_n$, $\Omega_n \subset \Omega_{n+1}$ (This clearly shows that

\mathcal{G} is not finitely presented). The subgroups Ω_n can be defined recursively as follows:

$$\Omega_1 = \text{Ker}(\bar{\varphi}) \quad \text{and} \quad \Omega_n = \{w \in \Xi \mid w_0, w_1 \in \Omega_{n-1}\}$$

It is known that $\Omega_n = \langle u_1, \dots, u_n, v_0, \dots, v_{n-1} \rangle_\Gamma^\#$ (see [9]).

The subgroups Ω_n are related to the "branch algorithm" which solves the word problem in \mathcal{G} (See [13]). Roughly speaking Ω_n consists of elements for which the algorithm stops after n steps.

Similarly we have subgroups Υ_n of Γ such that $\mathcal{G}_n = \mathcal{G}/St_{\mathcal{G}}(n) = \Gamma/\Upsilon_n$ where $\Upsilon_{n+1} \subset \Upsilon_n$ and $\bigcap_n \Upsilon_n = \Omega$. Hence $St_{\mathcal{G}}(n) = \Upsilon_n/\Omega$. A recursive definition for Υ_n is:

$$\Upsilon_1 = \Xi$$

and

$$\Upsilon_n = \{w \in \Xi \mid w_0, w_1 \in \Upsilon_{n-1}\}$$

We will prove theorem 1 by showing that for $n \geq 3$ we have

$$\Upsilon_n = \langle u_1, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle_\Gamma^\#$$

This will be done by induction on n and the case $n = 3$ follows from the following 3 lemmas:

Lemma 1. *We have*

$$\mathcal{G}_3 \cong (C_2 \wr C_2) \wr C_2 = \langle x, y, z \mid x^2, y^2, z^2, [x, x^y], [y, y^z], [x, x^z], [x, y^z], [y, x^z] \rangle \quad (3)$$

where $x = ada, y = c, z = a$

Proof. Direct inspection of the action of a, c, ada on the tree consisting of the first 3 levels (See [7] page 226). \square

Lemma 2. *Presentation (3) is equivalent to*

$$\mathcal{G}_3 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^4, (adac)^4 \rangle \quad (4)$$

Proof. Follows from the following equations and applying the Tietze transformations to (3).

$$\begin{aligned}
[x, x^y] &= [ada, cadac] = (adac)^4 \\
[y, y^z] &= [c, aca] = (ca)^4 \\
[x, x^z] &= [ada, d] = (ad)^4 \\
[x, y^z] &= [ada, aca] \\
[y, x^z] &= [c, d]
\end{aligned}$$

□

Lemma 3. *Presentation (4) is equivalent to*

$$\mathcal{G}_3 = \langle a, b, c, d \mid a^2, b^2, c^2, d^2, bcd, (ad)^4, (ac)^4, (abac)^4 \rangle \quad (5)$$

and hence $\Upsilon_3 = \langle w_3, t_3 \rangle_\Gamma^\#$.

Proof. Follows from the following equalities:

$$(adac)^4 = (adacadac)^2 = (adcacacdac)^2 = (abacabac)^2 = (abac)^4$$

where in step 3 we use the equality $aca = cacac$. □

Recall that \mathcal{G} is regular branch over the subgroup $K = \langle (ab)^2 \rangle_\mathcal{G}^\#$.

Lemma 4. $St_\mathcal{G}(3) \leq K$ and hence

$$\varphi(St_\mathcal{G}(n)) = St_\mathcal{G}(n-1) \times St_\mathcal{G}(n-1)$$

for $n \geq 4$, therefore we have

$$\bar{\varphi}(\Upsilon_n) = \Upsilon_{n-1} \times \Upsilon_{n-1}$$

for $n \geq 4$.

Proof. The fact that $St_\mathcal{G}(3) \leq K$ is proven in [7] (page 230). Therefore, since \mathcal{G} is a regular branch group over K (i.e $K \times K \leq \psi(K)$) we get the remaining equalities. □

Proof of theorem 1. We have to show that for $n \geq 3$ we have

$$\Upsilon_n = \langle u_1, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle_\Gamma^\#$$

Now induction on n , equations (2) and the fact $\bar{\varphi}(\Upsilon_n) = \Upsilon_{n-1} \times \Upsilon_{n-1}$ show that

$$\Upsilon_n = Ker(\bar{\varphi}) \langle u_2, \dots, u_{n-3}, v_1, \dots, v_{n-4}, w_n, t_n \rangle_\Gamma^\#$$

but $Ker(\bar{\varphi}) = \Omega_1 = \langle u_1, v_0 \rangle_\Gamma^\#$ from which we obtain

$$\Upsilon_n = \langle u_1, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle_\Gamma^\#$$

3.2 Computation of Schur Multiplier of \mathcal{G}_n

This section is devoted to the proofs of theorems 2 and 3. The ideas are analogous to [10] with slight modifications where needed.

Let F be the free group on $\{a, b, c, d\}$ and let

$$K_n = \langle a^2, b^2, c^2, d^2, bcd, u_0, \dots, u_{n-3}, v_0, \dots, v_{n-4}, w_n, t_n \rangle_F^\#$$

so that by theorem 1 we have $F/K_n \cong \mathcal{G}_n$. As mentioned before, the Schur multiplier can be calculated using Hopf's formula by:

$$M(\mathcal{G}_n) \cong K_n \cap F' / [K_n, F]$$

We have the following basic fact which will be used in the remainder:

Lemma 5. *We have the following inclusions:*

$$\begin{aligned} K_{n+1} &\subset K_n \\ \sigma(K_n) &\subset K_n \\ \sigma([K_n, F]) &\subset [K_n, F] \end{aligned}$$

Proof. The first inclusion follows from the fact that \mathcal{G}_{n+1} maps onto \mathcal{G}_n . The images of generators of K_n clearly lie in K_{n+1} and hence the second inclusion follows from the first one. Finally the third inclusion follows directly from the second. □

For computational reasons we need to change the relators in presentation of theorem 1 slightly to the ones given in the next lemma. The rationale behind this will be apparent when we will do computations modulo the subgroup $[K_n, F]$.

Lemma 6.

$$K_n = \langle B_1, B_2, B_3, B_4, L, U_0, \dots, U_{n-3}, V_0, \dots, V_{n-4}, W_n, T_n \rangle_F^\#$$

where

$$B_1 = a^2, B_2 = b^2, B_3 = c^2, B_4 = bcd$$

$$L = b^2 c^2 d^2 (bcd)^{-2}$$

$$U_i = \sigma^i((ad)^4 a^{-4} c^{-4}), V_i = \sigma^i((adacac)^4 a^{-12} c^{-8} d^{-4})$$

$$W_n = \sigma^{n-3}((ac)^4 a^{-4} c^{-4}), T_n = \sigma^{n-3}((abac)^4 a^{-8} b^{-4} c^{-4})$$

Proof. Let K'_n be the subgroup in the lemma. Clearly $B_i, L \in K_n$. Also since $U_0, V_0 \in K_n$ and $\sigma(K_n) \subset K_n$ we see that U_i, V_i are elements of K_n . We have

$$W_n = w_n \sigma^{n-3}(a^{-4} c^{-4}) \in K_n$$

similarly $T_n \in K_n$.

The converse inclusion can be shown similarly using $\sigma(K'_n) \subset K'_n$. \square

Let \approx denote equivalence modulo $[K_n, F]$.

Lemma 7. *In the group $K_n/[K_n, F]$ we have the equalities*

$$L^2 \approx U_i^2 \approx V_i^2 \approx W_n^2 \approx T_n^2 \approx 1$$

Proof. Observe that $x^2, [x, y] \in K_n$ where $x, y \in \{b, c, d\}^\pm$. Also

$$1 \approx [x^2, y] = [x, y]^x [x, y] \approx [x, y]^2$$

hence

$$\begin{aligned} L &= b^2 c^2 d^2 d^{-1} c^{-1} b^{-1} d^{-1} c^{-1} b^{-1} \approx dcb d^{-1} c^{-1} b^{-1} \approx dcb[d, c] c^{-1} d^{-1} b^{-1} \\ &\approx [d, c] dcb c^{-1} [d, b] b^{-1} d^{-1} \approx [d, c] [d, b] d [c, b] d^{-1} \approx [d, c] [d, b] [c, b] \end{aligned}$$

therefore $L^2 \approx 1$.

$$\begin{aligned} U_0^2 &= (ad)^4 a^{-4} d^{-4} (ad)^4 a^{-4} d^{-4} \\ &\approx a^{-1} d^{-1} a^{-1} d^{-1} a^{-1} d^{-1} a^{-1} d^{-1} a^{-1} aadadadad \\ &= [a, (ad)^4] \approx 1 \end{aligned}$$

V_0 is conjugate via $adac$ to the product:

$$\begin{aligned} acabacacabacacabacaca^{-12} c^{-8} d^{-4} abacacabacacabacacabacaca^{-12} c^{-8} d^{-4} abac &\approx \\ a^{-1} c^{-1} a^{-1} d^{-1} a^{-1} c^{-1} a^{-1} c^{-1} a^{-1} d^{-1} a^{-1} c^{-1} a^{-1} c^{-1} a^{-1} d^{-1} a^{-1} c^{-1} a^{-1} c^{-1} a^{-1} & \\ d^{-1} a^{-1} c^{-1} a^{-1} aacabacaacabacaacabacaacabac & \\ = [a, (acadac)^4] \approx 1 & \end{aligned}$$

Where the last equality is true since $(acadac)^4$ is a conjugate of $(adacac)^4 \in K_n$.

For W_n , a similar calculation (like the one for U_0) gives the following:

$$W_n^2 \approx [\sigma^{n-3}(a), \sigma^{n-3}((ac)^4)] \approx 1$$

Also similar computation (like the one for V_0) shows that T_n^2 is conjugate to

$$[\sigma^{n-3}(d), \sigma^{n-3}((baca)^4)] \approx 1$$

□

Lemma 8. *In $K_n/[K_n, F]$ we have $\langle B_1, B_2, B_3, B_4 \rangle \cong C^4$ where C denotes the infinite cyclic group.*

Proof. We have the quotient map

$$K_n/[K_n, F] \longrightarrow K_n/(K_n \cap F')$$

and the right hand side is a free abelian group since

$$K_n/(K_n \cap F') \cong K_n F' / F' \leq F / F' \cong C^4$$

Now B_1, B_2, B_3, B_4 are mapped onto the vectors

$$(2, 0, 0, 0), (0, 2, 0, 0), (0, 0, 2, 0), (0, 1, 1, 1)$$

respectively. Linear independence of these vectors proves the assertion. □

Lemma 9. *We have the following isomorphism:*

$$K_n/[K_n, F] \cong C^4 \times M_n$$

where C^4 is freely generated by B_1, B_2, B_3, B_4 and is isomorphic to $K_n/(K_n \cap F')$

and M_n is the torsion part generated by $\{L, U_i, V_i, W_n, T_n\}$, which is an elementary abelian 2-group isomorphic to $(K_n \cap F')/[K_n, F]$.

Proof. We have the split exact sequence

$$1 \rightarrow (K_n \cap F')/[K_n, F] \rightarrow K_n/[K_n, F] \rightarrow K_n/(K_n \cap F') \rightarrow 1$$

From previous lemma we have $K_n/(K_n \cap F') \cong C^4$. Hence

$$M_n \cong (K_n \cap F')/[K_n, F]$$

and is elementary abelian 2-group by lemma 6. □

Lemma 10. *The elements L, U_0, W_3, T_3 are independent in M_3 .*

Proof. Clearly M_3 maps to the abelian group

$$Q_3 = F' / ([K_3, F] \gamma_5(F) F^{(2)})$$

The result follows from the next lemma. □

Lemma 11. *Q_3 has the following presentation:*

Generators:

- $[a, b], [a, c], [a, d], [b, c]$
- $[a, x, y], \quad x \neq y, \quad x, y \in \{b, c, d\}$
- $[a, x, y, z], \quad x \neq y, \quad y \neq z \text{ and } (x, y, z) \text{ is not a permutation of } (b, c, d)$
where $x, y \in \{b, c, d\}$ and $z \in \{a, b, c, d\}$

Relations:

- *commutativity relations*
- $[a, b]^8 = [a, c]^4 = [a, d]^4 = [b, c]^2 = 1$
- $[a, b, c]^4 = [a, b, d]^4 = [a, c, b]^2 = [a, c, d]^2 = [a, d, b]^2 = [a, d, c]^2 = 1$
- $[a, x, y, z]^2 = 1$

Moreover the the images of L, U_0, W_3, T_3 in Q_3 are $[b, c], [a, d]^2, [a, c]^2, [a, b, c]^{-2}$ respectively.

Proof. (Throughout this proof \approx denotes equivalence modulo $([K_3, F] \gamma_5(F) F^{(2)})$)

Since $a^2, b^2, c^2, d^2 \in K_3$, using standard commutator calculus and the fact that $\gamma_5(F)$ appears in the denominator of Q_3 , it is easy to see that Q_3 is generated by elements of the form

- $[x, y], \quad x \neq y. \quad x, y \in \{a, b, c, d\}$
- $[x, y, z], \quad x \neq y, x, y, z \in \{a, b, c, d\}$
- $[x, y, z, w]. \quad x \neq y, \quad x, y, z, w \in \{a, b, c, d\}$

Before beginning calculations, we wish to write two equalities which will be frequently used in the remainder:

$$[x, yz] = [x, z][x, y][x, y, z] \quad (6)$$

$$[xy, z] = [x, z][x, y, z][y, z] \quad (7)$$

Clearly, in Q_3 we have the following relations:

$$[x, B_i] = [x, L] = [x, U_0] = [x, W_3] = [x, T_3] = 1 \quad x \in \{a, b, c, d\}$$

Using these we will further reduce the system of generators.

Firstly, from equation (6) we have:

$$[x, a]^2[x, a, a] = [x, a^2] \approx 1, \quad x \in \{b, c, d\}$$

Hence

$$[x, a, a] \approx [a, x]^2 \quad (8)$$

and we can omit the generators $[x, a, a]$ where $x \in \{b, c, d\}$.

Since

$$[x, a, y] = [a, x][y, [a, x]][x, a] \approx [y, [a, x]] = [a, x, y]^{-1} \quad (9)$$

we also can omit generators $[a, x, a]$ and $[x, a, y]$ where $x, y \in \{b, c, d\}$.

Next, again using equation (6) we have

$$[x, y]^2[x, y, y] = [x, y^2] \approx 1 \quad x \in \{a, b, c, d\}, y \in \{b, c, d\}$$

Hence

$$[x, y, y] \approx [y, x]^2 \quad (10)$$

and therefore the generators $[a, y, y]$ where $y \in \{b, c, d\}$ can be omitted.

Since $[x, y] \in K_3$ for $x, y \in \{b, c, d\}$, we also omit generators of the form $[x, y, a]$.

Using equations (9), (8) and (7) we have

$$[a, x, a, y] \approx [[x, a, a]^{-1}, y] \approx [[a, x]^{-2}, y] \approx [[a, x]^{-1}, y]^2 = [x, a, y]^2$$

which enables us to omit generators of the form $[a, x, a, y]$ where $x \in \{b, c, d\}$ and $y \in \{a, b, c, d\}$.

Similarly, using equations (10) and (7) we have

$$[a, x, x, z] \approx [[x, a]^2, z] \approx [[x, a], z]^2 = [x, a, z]^2$$

which enables us to omit generators of the form $[a, x, x, z]$ where $x \in \{b, c, d\}$ and where $z \in \{a, b, c, d\}$.

The following equation holds:

$$\begin{aligned} [x, yzt] &= [x, zt][x, y][x, y, zt] \\ &= [x, t][x, z][x, z, t][x, y][x, y, t][x, y, z][x, y, z, t] \end{aligned} \quad (11)$$

Substituting $x = a$ yields the omission of the generators of the form $[a, y, z, t]$ where (y, z, t) is a permutation of (b, c, d) .

Similarly, substituting different letters into equation (11) we get the following identities:

$$1 \approx [b, bcd] \approx [b, c][b, d]$$

$$1 \approx [c, bcd] \approx [c, d][c, b]$$

$$1 \approx [d, bcd] \approx [d, b][d, c]$$

Which yield the identities $[c, d] \approx [b, c] \approx [d, b]$. Thus $[c, d]$ and $[b, d]$ can be omitted from the system of generators.

Hence Q_3 has the asserted set of generators. We proceed to showing it has the given relators.

Using equation (10) we get $[b, c]^2 \approx 1$.

Using similar calculations as before we get :

$$1 \approx [a, (ad)^4] \approx [a, d]^4$$

$$1 \approx [a, (ac)^4] \approx [a, c]^4$$

$$1 \approx [a, (ab)^8] \approx [a, b]^8$$

We have

$$1 \approx [x, (ad)^4] = [x, [a, d]^2] = [x, [a, d]]^2 [[x, ad], [a, d]] \approx [x, [a, d]]^2 \approx [a, d, x]^{-2}$$

$$\text{hence } [a, d, b]^2 = [a, d, c]^2 \approx 1$$

Similarly

$$1 \approx [x, (ac)^4] \approx [a, c, x]^{-2}$$

$$1 \approx [x, (ab)^8] \approx [a, b, x]^{-4}$$

$$\text{yield } [a, c, b]^2 \approx [a, c, d]^2 \approx 1 \text{ and } [a, b, c]^4 \approx [a, b, d]^4 \approx 1$$

Finally

$$1 \approx [a, x, y, z^2] \approx [a, x, y, z]^2 [a, x, y, z, z] \approx [a, x, y, z]^2$$

where $x, y \in \{b, c, d\}$ and $a \in \{a, b, c, d\}$.

Let us show that the images of L, U_0, W_3, T_3 in Q_3 are $[b, c], [a, d]^2, [a, c]^2, [a, b, c]^{-2}$ respectively:

By Lemma 6, $L \approx [d, c][d, b][c, b] \approx [b, c]$

Also similar to earlier computations we have;

$$U_0 \approx [a, d]^2$$

$$W_3 \approx [a, c]^2$$

$$\begin{aligned} T_3 &= (abac)^4 a^{-8} b^{-4} c^{-4} = abacabacabacabacabaca^{-8} b^{-4} c^{-4} \\ &\approx [a, b] b^{-1} c^{-1} a^{-1} bac [a, b] b^{-1} c^{-1} a^{-1} bac \\ &= ([a, b][b, ac])^2 \\ &= ([a, b][b, c][b, a][b, a, c])^2 \\ &\approx [b, a, c]^2 \approx [a, b, c]^{-2} \text{ (by equation 9)} \end{aligned}$$

Therefore

$$1 \approx [a, T_3] \approx [a, [a, b, c]^{-2}] \approx [a, [a, b, c]]^{-2} \approx [a, b, c, a]^2$$

Similarly

$$1 \approx [x, T_3] \approx [x, b, c, a]^2$$

Now the following argument finishes the proof of the lemma:

It is clear that all relations of Q_3 can be derived from the relations

$$[x, B_i] = [x, L] = [x, U_0] = [x, W_3] = [x, T_3] = 1, x \in \{a, b, c, d\}$$

together with relations of the form $y = 1$ where $y \in \gamma_5(F)$ or $y \in F^{(2)}$.

The relations $y \in \gamma_5(F)$ can be disregarded by omitting the generators of commutator length 5 or more from the generating system. The relations $y \in F^{(2)}$ translate to commutativity relations among generators. Finally above computations show that one can further reduce the generating set to the one asserted in the lemma. Also the calculations imply that the relations are equivalent to the system of relators given in the lemma. Hence Q_3 has the given presentation, and clearly L, U_0, W_3, T_3 are independent in Q_3 . \square

Lemma 12. *The elements L, U_0, U_1, V_0 are independent in M_n where $n \geq 4$.*

Proof. M_n maps to the abelian group $Q_n = F'/([K_n, F]\gamma_5(F)F^{(2)})$. The result is a corollary of the next lemma. \square

Lemma 13. *Q_n has the following presentation:*

Generators:

- $[a, b], [a, c], [a, d], [b, c]$
- $[a, x, y], \quad x \neq y, \quad x, y \in \{b, c, d\}$
- $[a, x, y, z], \quad x \neq y, \quad y \neq z \text{ and } (x, y, z) \text{ is not a permutation of } (b, c, d)$
where $x, y \in \{b, c, d\}$ and $z \in \{a, b, c, d\}$

Relations:

- *commutativity relations*
- $[a, b]^{16} = [a, c]^8 = [a, d]^4 = [b, c]^2 = 1$

- $[a, b, c]^8 = [a, b, d]^8 = [a, c, b]^4 = [a, c, d]^4 = [a, d, b]^2 = [a, d, c]^2 = 1$
- $[a, x, y, z]^2 = 1$

Moreover the the images of L, U_0, U_1, V_0 in Q_n are $[b, c], [a, d]^2, [a, c]^4, [a, d]^2[a, c, d]^2$ respectively.

Proof. Most of the proof is similar to the proof of Lemma (11). Additionally we only need to show that the relations:

$$[x, U_1] = \dots = [x, U_{n-3}] = [x, V_0] = \dots = [x, V_{n-4}] = [x, W_n] = [x, T_n] = 1$$

are consequences of the given system of relators.

Let \cong mean equality in F' modulo the subgroup $[K_n, F]F^{(2)}$.

Using equation (10) we have:

$$[a, c]^2 \cong [a, c, c]^{-1}$$

Also using equation (6)

$$1 \cong [a, c, c^2] \cong [a, c, c]^2[a, c, c, c]$$

which implies

$$[a, c, c]^{-2} \cong [a, c, c, c]$$

and hence

$$U_1 \cong [a, c]^4 \cong [a, c, c]^{-2} \cong [a, c, c, c]$$

which yields $U_1 \in \gamma_4(F) \bmod [K_n, F]F^{(2)}$. Therefore $[x, U_1] \in \gamma_5(F) \bmod [K_n, F]F^{(2)}$.

It follows that relations of the form $[x, U_1]$ are consequences of previous relations. Since $\sigma(\gamma_5(F)) \leq \gamma_5(F)$ and $\sigma(F^{(2)}) \leq F^{(2)}$, we also see that relations of the form $[x, U_i]$ where $i = 2, \dots, n-3$ are consequences of previous relations.

For V_0 we have:

$$\begin{aligned}
V_0 &= (adacac)^4 a^{-12} c^{-8} d^{-4} \\
&= (ad)^2 (ac)^2 [(ac)^2, ad] (ac)^2 (ad)^2 (ac)^2 [(ac)^2, ad] (ac)^2 a^{-12} c^{-8} d^{-4} \\
&\approx [a, d]^2 [a, c]^4 [[a, c], ad]^2 \\
&\approx [a, d]^2 [a, c]^4 [a, c, d]^2 [a, c, a]^2 [a, c, a, d]^2 \\
&\approx [a, d]^2 [a, c]^4 [a, c, d]^2 [c, a, a]^{-2} [a, c, a, d]^2 \\
&\approx [a, d]^2 [a, c]^4 [a, c, d]^2 [a, c]^4 [a, c, a, d]^2 \\
&\approx [a, d]^2 [a, c, d]^2
\end{aligned}$$

hence

$$[x, V_0] \approx [x, [a, d]^2 [a, c, d]^2] \approx [x, [a, d]]^2 [a, [a, c, d]]^2 = [a, d, x]^2 [a, c, d, x]^2$$

is a consequence of previous relations.

For V_1 we get:

$$\begin{aligned}
V_1 &= (acacacabacab)^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong ((ac)^4 ab [ab, ac] [[ab, ac], ab])^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong (ac)^{16} (ab)^8 [[ab, ac], ab]^4 (aca)^{-12} b^{-8} c^{-4} \\
&\cong [c, a, a, a, a] [b, a, a, a]
\end{aligned}$$

therefore $V_1 \in \gamma_4(F) \bmod [K_n, F] F^{(2)}$ hence relations of the form $[x, V_i]$ where $i = 1, \dots, n-4$ are consequences of previous relations.

$$\begin{aligned}
W_4 &= (acab)^4 (aca)^{-4} b^{-4} \\
&\cong ([a, c] [c, ab])^2 \\
&\cong [a, c]^2 ([c, b] [c, a] [c, a, b])^2 \\
&\cong [c, a, b]^2
\end{aligned}$$

But by equation (6),

$$1 \cong [c, a, b^2] \cong [c, a, b]^2 [c, a, b, b]$$

therefore

$$W_4 \cong [c, a, b, b]^{-1}$$

hence $W_n = \sigma^{n-4}(W_4) \in \gamma_4(F)$ modulo $[K_n, F]F^{(2)}$ and relations $[x, W_n]$ follow from the previous relations.

Finally for T_3 from previous computations we get:

$$T_3 \cong [a, b, c]^{-2}$$

using

$$1 \cong [[a, b], c^2] \cong [a, b, c]^2[a, b, c, c]$$

we get

$$T_3 \cong [a, b, c]^{-2} \cong [a, b, c, c]$$

hence $T_n = \sigma^{n-3}(T_3) \in \gamma_4(F)$ modulo $[K_n, F]F^{(2)}$ and relations $[x, T_n]$ follow from the previous relations. □

Let $P = \langle b, c, d, b^a, c^a, d^a \rangle_F$ be the lift of $St_{\mathcal{G}}(1)$ to F and let

$$\psi : P \rightarrow \Gamma \times \Gamma$$

be the homomorphism similar to $\bar{\varphi}$.

Lemma 14. *We have $Ker(\psi) = \langle B_1, B_2, B_3, B_4, L, U_0, U_1, V_0 \rangle_F^\#$*

Proof. Restatement of [10] lemma 11. □

Lemma 15. *For $n \geq 4$ the following isomorphism holds:*

$$K_n/([K_n, F]) \cong C^4 \times C_2^4 \times \psi(K_n)/\psi([K_n, F])$$

where the factor C^4 is generated by B_1, \dots, B_4 and C_2^4 is generated by L, U_0, U_1, V_0 .

Proof. Let

$$\psi_* : K_n/[K_n, F] \rightarrow \psi(K_n)/\psi([K_n, F])$$

be the homomorphism induced by ψ .

Then by lemma (14) we have:

$$\begin{aligned} Ker(\psi_*) &= (Ker(\psi))[K_n, F]/[K_n, F] \\ &= \langle B_1, B_2, B_3, B_4, L, U_0, U_1, V_0 \rangle_{K_n/[K_n, F]} \\ &\cong C^4 \times C_2^4 \end{aligned}$$

Hence from lemma (9) the result follows. □

Lemma 16. *We have:*

$$\begin{aligned}\psi(U_i) &= (1, U_{i-1}) \\ \psi(V_i) &= (1, V_{i-1}) \\ \psi(W_n) &= (1, W_{n-1}) \\ \psi(T_n) &= (1, T_{n-1})\end{aligned}$$

for $i \geq 1$ and $n \geq 4$.

Proof. We have

$$\psi\sigma \begin{cases} a \mapsto (d, a) \\ b \mapsto (1, b) \\ c \mapsto (a, c) \\ d \mapsto (a, d) \end{cases}$$

Hence the image of $\pi_1\psi\sigma$ lies in the subgroup $\langle a, d \rangle_\Gamma$ and $\pi_2\psi\sigma$ is the identity map. Since $\langle a, d \rangle_\Gamma$ has exponent 4 we get the asserted equalities. \square

Let

$$\Theta_n = \langle U_1, \dots, U_{n-3}, V_0, \dots, V_{n-4}, W_n, T_n \rangle_\Gamma^\#$$

so that $\Gamma/\Theta_n \cong \mathcal{G}_n$.

Lemma 17. *The following relations hold:*

$$\psi(K_n) = \Theta_{n-1} \times \Theta_{n-1}$$

$$\psi([K_n, F]) = ([\Theta_{n-1}, \Gamma] \times [\Theta_{n-1}, \Gamma])\Psi$$

where $\Psi \leq \Gamma \times \Gamma$ is the subgroup consisting of elements of the form (w^{-1}, w) , $w \in \Theta_{n-1}$.

Proof. Similar to [10] lemma 14. \square

Lemma 18. *We have the isomorphism:*

$$\psi(K_n)/\psi([K_n, F]) \cong \Theta_{n-1}/([\Theta_{n-1}, \Gamma])$$

and the generators $\psi(U_i), \psi(V_i), \psi(W_n), \psi(T_n)$ are mapped to the generators $U_{i-1}, V_{i-1}, W_{n-1}, T_{n-1}$ respectively.

Proof. By lemma 17, $\psi(K_n)/\psi([K_n, F])$ is isomorphic to

$$(\Theta_{n-1} \times \Theta_{n-1})/(([\Theta_{n-1}, \Gamma] \times [\Theta_{n-1}, \Gamma])\Psi) \quad (12)$$

Since $(1, x)^{-1}(x, 1) = (x, x^{-1}) \in \Xi$, (12) is generated by elements of the form $(1, x)$ where $x \in \{U_1, \dots, U_{n-4}, V_0, \dots, V_{n-5}, W_{n-1}, T_{n-1}\}$.

It is easy to check that the map

$$(1, x) \mapsto x$$

gives an isomorphism between (12) and $\Theta_{n-1}/([\Theta_{n-1}, \Gamma])$. \square

Lemma 19. W_3, T_3 are independent in $\Theta_3/([\Theta_3, \Gamma])$

Proof. The proof is analogous to the proof of lemma (11) and is omitted. \square

Lemma 20. For $n \geq 4$, U_1, V_0 are independent in $\Theta_n/([\Theta_n, \Gamma])$

Proof. The proof is analogous to the proof of lemma (12) and is omitted. \square

Lemma 21. For $n \geq 4$ we have

$$\Theta_n/([\Theta_n, \Gamma]) \cong C_2^2 \times \Theta_{n-1}/([\Theta_{n-1}, \Gamma])$$

Where U_1, V_0 are generators of the factor C_2^2 and the images of elements $U_2, \dots, U_{n-3}, V_1, \dots, V_{n-4}, W_n, T_n$ are generators of the second factor.

Proof. Similar to the proof of lemma (15) using lemma (20). \square

Proof of Theorem 2. We need to show $n \geq 3$, we have

$$H_2(\mathcal{G}_n, \mathbb{Z}) \cong (C_2)^{2n-2}$$

We claim that for $n \geq 3$

$$\Theta_n/([\Theta_n, \Gamma]) \cong C_2^{2n-4}$$

The case $n = 3$ follows from lemma (19). Assume it holds for $n > 3$. Then by lemma (21)

$$\Theta_{n+1}/([\Theta_{n+1}, \Gamma]) \cong C_2^2 \times \Theta_n/([\Theta_n, \Gamma])$$

and the claim follows from the induction hypothesis. Hence

$$K_n/([K_n, F]) \cong C^4 \times C_2^{2n-2}$$

and the result follows from lemma (9) and Hopf's formula.

Proof of theorem 3. Let $\bar{K}_n = K_n/[K_n, F]$ and $\dot{K} = \bar{K}_n / \langle B_1^2, B_2^2, B_3^2, B_4^2 \rangle$. We have the following homomorphism:

$$\begin{array}{ccc} \bar{K}_n & \longrightarrow & \dot{K}_n \\ a^2 & \mapsto & B_1 \\ b^2 & \mapsto & B_2 \\ c^2 & \mapsto & B_3 \\ d^2 & \mapsto & B_3 B_2 L \\ u_i & \mapsto & U_i \\ v_i & \mapsto & V_i \\ w_n & \mapsto & W_n \\ t_n & \mapsto & T_n \end{array}$$

Hence any dependence among the initial relators will produce dependence among generators of \dot{K}_n .

3.3 Computation of the cohomology group $H^2(\hat{\mathcal{G}}, \mathbb{F}_2)$

This subsection is devoted to the proof of theorem 5.

It is well known (see for example [17]) that for a finite abelian group A one has

$$H^2(G, A) \cong (G/G' \otimes A) \times (M(G) \otimes A)$$

Using this we have :

Lemma 22. For $n \geq 3$ $H^2(\mathcal{G}_n, \mathbb{F}_2) \cong C_2^{2n+1}$.

Proof. As mentioned before we have $\mathcal{G}_n/\mathcal{G}'_n \cong (C_2)^3$. Since $M(\mathcal{G}_n) \cong C_2^{2n-2}$ and $C_2 \otimes C_2 \cong C_2$ it follows that

$$H^2(\mathcal{G}_n, \mathbb{F}_2) \cong C_2^{2n+1}$$

□

Lemma 23. *For natural numbers n, k with $n \geq 3$ let*

$$q_{n,k} : \mathcal{G}_{n+k} \rightarrow \mathcal{G}_n$$

be the canonical quotient map. Then there is $N \in \mathbb{N}$ such that for all n, k the dimension of the kernel of the induced map

$$q_{n,k}^* : H^2(\mathcal{G}_n, \mathbb{F}_2) \rightarrow H^2(\mathcal{G}_{n+k}, \mathbb{F}_2)$$

is bounded above by N .

Proof. We have an exact sequence

$$1 \rightarrow \text{Ker}(q_{n,k}) \rightarrow \mathcal{G}_{n+k} \rightarrow \mathcal{G}_n \rightarrow 1 \quad (13)$$

and clearly $\text{Ker}(q_{n,k}) \cong \text{St}_{\mathcal{G}}(n)/\text{St}_{\mathcal{G}}(n+k)$.

The sequence (13) induces the five term exact sequence (See [21])

$$\begin{array}{ccccccc} 0 & \rightarrow & \text{Hom}(\mathcal{G}_n, \mathbb{F}_2) & \xrightarrow{\alpha} & \text{Hom}(\mathcal{G}_{n+k}, \mathbb{F}_2) & \xrightarrow{\beta} & \text{Hom}(\text{Ker}(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n} \\ & & & \searrow \partial & & \searrow q_{n,k}^* & \\ & & & H^2(\mathcal{G}_n, \mathbb{F}_2) & & H^2(\mathcal{G}_{n+k}, \mathbb{F}_2) & \end{array}$$

where $\text{Hom}(\text{Ker}(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is the set of all homomorphisms invariant under the action of \mathcal{G}_n on $\text{Ker}(q_{n,k})$ by conjugation.

Since \mathcal{G}_n and \mathcal{G}_{n+k} have the same abelianization α is an isomorphism. Therefore β is the zero map and hence ∂ is an injection.

As noted earlier for $n \geq 4$ we have $\text{St}_{\mathcal{G}}(n) \cong \text{St}_{\mathcal{G}}(n-1) \times \text{St}_{\mathcal{G}}(n-1)$. Therefore we have

$$\text{Ker}(q_{n,k}) \cong \frac{\text{St}_{\mathcal{G}}(n)}{\text{St}_{\mathcal{G}}(n+k)} \cong \frac{\text{St}_{\mathcal{G}}(3)}{\text{St}_{\mathcal{G}}(k+3)} \times \cdots \times \frac{\text{St}_{\mathcal{G}}(3)}{\text{St}_{\mathcal{G}}(k+3)}$$

Since \mathcal{G}_n acts transitively on these factors any homomorphism in $\text{Hom}(\text{Ker}(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is uniquely determined by its values in the first factor of this decomposition. But $\text{St}_{\mathcal{G}}(3)$ is finitely generated hence the dimension of $\text{Hom}(\text{Ker}(q_{n,k}), \mathbb{F}_2)^{\mathcal{G}_n}$ is no more than a fixed number and in particular independent of k and n . \square

Lemma 24. *Suppose $\{G_i, \varphi_{ij} \mid i, j \in I\}$ is a direct system of finitely generated elementary abelian p -groups. Suppose also that the sequence $\dim(G_i)$ is monotone increasing and there is a uniform bound N such that $\dim(\text{Ker}(\varphi_{ij})) \leq N$. Then the direct limit $\varinjlim G_i$ is infinite and hence isomorphic to $(C_p)^\infty$.*

Proof. Recall that the direct limit can be defined as the disjoint union $\bigsqcup G_i$ factored by the equivalence relation:

$$g_i \sim g_j \iff \exists k \geq i, j \text{ such that } \varphi_{ik}(g_i) = \varphi_{jk}(g_j)$$

Suppose that $\varinjlim G_i$ has M elements. Select i large enough so that $\dim(G_i) > NM$. So for large j we have

$$|G_i : \text{Ker}(\varphi_{ij})| \geq \frac{\dim(G_i)}{N} > \frac{NM}{N} = M$$

which shows that the direct limit has more than M elements. □

Proof of theorem 5: The following is well known: (See [22] page 178) If G is the inverse limit of the inverse system $\{G_i, \phi_{ij}\}$ then

$$H^n(G, \mathbb{F}_p) \cong \varinjlim H^n(G_i, \mathbb{F}_p)$$

i.e. $H^n(G, \mathbb{F}_p)$ is the direct limit of the direct system $\{H^n(G_i, \mathbb{F}_p), \phi_{ij}^*\}$ where ϕ_{ij}^* is the inflation map induced by ϕ_{ij} . Hence

$$H^2(\hat{\mathcal{G}}, \mathbb{F}_2) \cong \varinjlim H^2(\mathcal{G}_n, \mathbb{F}_2)$$

Now lemmas (22) and (23) show that the hypotheses of lemma (24) are satisfied and therefore $H^2(\hat{\mathcal{G}}, \mathbb{F}_2) \cong (C_2)^\infty$.

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